## Homework 3

## Due: Thursday 2/14/19 by 12:00pm (noon)

## Grading Scheme:

- 1 point for incorrect plots and answers to causality/invertibility.
- 2 points for correct plots and answers to causality/invertibility but no interpretation.
- 3 points for correct plots and answers to causality/invertibility but incorrect interpretations.
- 4 points for mostly everything correct.
- 5 points for nearly everything correct.

The R library polynom lets us easily compute the roots of polynomials. You'll need to install the polynom library and load it. I'll ask that you use it a bit in this homework, so here's a little example:

```
library(polynom)
# Create a "polynomial" object for the polynomial
# 1-5x+3x^2 + 2x-3
pol <- polynomial(c(1, -5, 3, 2))
# Get the values of x for which 1-5x+3x^2 + 2x^3 = 0
sol <- solve(pol)
```

You may get complex roots $r=a+b i$. Note that the absolute value of a complex number $r$ is given by $|r|=\sqrt{a^{2}+b^{2}}$.

1. Consider the following $\mathbf{A R}(p)$ models, all with $\sigma_{w}^{2}=1$.
i. $p=1, \phi_{1}=0.99$
ii. $p=2, \phi_{1}=0.04, \phi_{2}=0.92$
iii. $p=2, \phi_{1}=0.04, \phi_{2}=-0.92$
iv. $p=3, \phi_{1}=0.42, \phi_{2}=-0.29, \phi_{3}=0.15$
(a) For (i)-(iii), find the root of the autoregressive polynomial that is smallest in magnitude by solving $\phi(z)=0$ for $z$ by hand, without using any special R functions. For (iv), use polynom to find the root that is smallest in magnitude. Give the value of this root and indicate whether or not the model is causal.
```
pars <- list(c(0.99), c(0.04, 0.92), c(0.04, -0.92), c(0.42, -0.29, 0.15))
for (i in 1:length(pars)) {
    phi.z <- polynomial(c(1, -pars[[i]]))
    sol <- solve(phi.z)
}
```

i. $p=1, \phi_{1}=0.99$. We can solve this one easily by finding the value of $z$ that sets $1-\phi_{1} z=0$ - I'm not going to do out the work here. The smallest root in magnitude is 1.01 , which has absolute value of 1.01 , so this model is causal.
ii. $p=2, \phi_{1}=0.04, \phi_{2}=0.92$. We can solve this one using the quadratic formula - I'm not going to do out the work here. The smallest root in magnitude is 1.02 , and it absolute value of 1.02 so this model is causal.
iii. $p=2, \phi_{1}=0.04, \phi_{2}=-0.92$. We can solve this one using the quadratic formula - I'm not going to do out the work here. Both roots have the same magnitude, they are equal to $0.02 \pm 1.04 i$. They have absolute value 1.04 , so this model is causal.
iv. $p=3, \phi_{1}=0.42, \phi_{2}=-0.29, \phi_{3}=0.15$. There three roots, and two with the smallest magnitude. These roots are equal to $-0.09 \pm 1.78 i$. They have absolute value 1.78 , so this model is causal.
(b) Using ARMAacf to compute $\rho_{x}(h)$, plot the autocorrelation function $\rho_{x}(h)$ for $h=0, \ldots, 10$ for the causal AR $(p)$ models on a single plot. Include a dotted horizontal line at 0 .

```
lag.max <- 10
acfs <- matrix(nrow = length(pars), ncol = lag.max + 1)
for (i in 1:length(pars)) {
    acfs[i, ] <- ARMAacf(ar = pars[[i]], lag.max = lag.max)
}
plot(0:lag.max, acfs[1, ], type = "n", ylim = range(acfs),
    ylab = expression(gamma[x](h)), xlab = "h")
for (i in 1:length(pars)) {
    points(0:lag.max, acfs[i, ], col = i, pch = 16)
    lines(0:lag.max, acfs[i, ], col = i)
}
abline(h = 0, lty = 3)
legend("bottomleft", col = 1:4, pch = rep(16, 4), legend = c("i", "ii", "iii", "iv"))
```


(c) Based on the plot you made in (b), describe what kinds of values of the autoregressive coefficients $\phi_{1}, \ldots, \phi_{p}$ produce the following behaviors, using at most one sentence for each:

- $\rho_{x}(h)$ oscillates between positive and negative values;
- $\rho_{x}(h)$ oscillates between larger and smaller but always positive values;
- $\rho_{x}(h)$ decays very quickly;
- $\rho_{x}(h)$ decays very slowly.
- Models iii. and iv. have ACF's that oscillate between positive and negative values. They are the only models that have a mix of positive and negative AR parameters, suggesting that a mix of signs can lead to autocorrelations that alternate between positive and negative.
- Model ii. has an ACF that oscillates between larger and smaller but always positive values. It is also one of the only models that has complex solutions to the AR polynomial, which suggests that AR models wtih complex solutions to their AR polynomials may have periodic/cyclic/seasonal behaviors.
- Model iv. is the only model with an ACF that decays very quickly. All of the other models have at least one AR parameter that is close to 1 in absolute value, which suggests that smaller values of the AR parameters can yield more rapidly decaying ACFs.
- Models i. and ii. have ACF's that decay slowly, and model iii. has an ACF that decays slowly in
absolute value. All of these models have at least one AR parameter that is close to 1 in absolute value, which suggests that larger values of the AR parameters can yield more slowly decaying ACFs.

2. Consider the following $\mathbf{M A}(q)$ models, all with $\sigma_{w}^{2}=1$.
i. $q=1, \theta_{1}=0.99$
ii. $q=2, \theta_{1}=0.04, \theta_{2}=0.92$
iii. $q=2, \theta_{1}=0.04, \theta_{2}=-0.92$
iv. $q=3, \theta_{1}=0.42, \theta_{2}=-0.29, \theta_{3}=0.15$
(a) For (i)-(iii), find the root of the moving average polynomial that is smallest in magnitude by solving $\theta(z)=0$ for $z$ by hand, without using any special R functions. For (iv), use polynom to find the root that is smallest in magnitude. Give the value of this root and indicate whether or not the model is invertible.
```
pars <- list(c(0.99), c(0.04, 0.92), c(0.04, -0.92), c(0.42, -0.29, 0.15))
for (i in 1:length(pars)) {
    theta.z <- polynomial(c(1, pars[[i]]))
    sol <- solve(theta.z)
}
```

i. $q=1, \theta_{1}=0.99$. We can solve this one easily by finding the value of $z$ that sets $1+\theta_{1} z=0$ - I'm not going to do out the work here. The smallest root in magnitude is -1.01 , which has absolute value of 1.01, so this model is invertible.
ii. $q=2, \theta_{1}=0.04, \theta_{2}=0.92$. We can solve this one using the quadratic formula - I'm not going to do out the work here. Both roots have the same magnitude, they are equal to $-0.02 \pm 1.04 i$. They have absolute value 1.04 , so this model is invertible.
iii. $q=2, \theta_{1}=0.04, \theta_{2}=-0.92$. We can solve this one using the quadratic formula - I'm not going to do out the work here. The smallest root in magnitude is -1.02 , and it absolute value of 1.02 so this model is causal.
iv. $q=3, \theta_{1}=0.42, \theta_{2}=-0.29, \theta_{3}=0.15$. The root that is smallest in magnitude is equal to 1.09 . They have absolute value 1.09 , so this model is invertible.
(b) Using ARMAacf to compute $\rho_{x}(h)$, plot the autocorrelation function $\rho_{x}(h)$ for $h=0, \ldots, 10$ for the invertible MA $(q)$ models on a single plot. Include a dotted horizontal line at 0 .

```
lag.max <- }1
acfs <- matrix(nrow = length(pars), ncol = lag.max + 1)
for (i in 1:length(pars)) {
    acfs[i, ] <- ARMAacf(ma = pars[[i]], lag.max = lag.max)
}
plot(0:lag.max, acfs[1, ], type = "n", ylim = range(acfs),
        ylab = expression(gamma[x](h)), xlab = "h")
for (i in 1:length(pars)) {
    points(0:lag.max, acfs[i, ], col = i, pch = 16)
    lines(0:lag.max, acfs[i, ], col = i)
}
abline(h = 0, lty = 3)
legend("topright", col = 1:4, pch = rep(16, 4), legend = c("i", "ii", "iii", "iv"))
```


(c) Based on the plot you made in (b) and the values you computed for the autocorrelation function $\rho_{x}(h)$, give the largest lag for which $\left|\rho_{x}(h)\right|$ is greater than zero for each MA $(q)$ model.
The autocorrelation $\left|\rho_{x}(h)\right|$ is always exactly equal to zero when $h>q$.
3. Consider the following $\operatorname{ARMA}(p, q)$ models, all with $\sigma_{w}^{2}=1$.
i. $p=1, q=1, \phi_{1}=0.99, \theta_{1}=0.99$
ii. $p=1, q=2, \phi_{1}=0.99, \theta_{1}=0.04, \theta_{2}=-0.92$
iii. $p=2, q=2, \phi_{1}=0.04, \phi_{2}=0.92, \theta_{1}=0.04, \theta_{2}=0.92$
iv. $p=2, q=2, \phi_{1}=0.04, \phi_{2}=-0.92, \theta_{1}=0.04, \theta_{2}=0.92$
v. $p=3, q=3, \phi_{1}=0.42, \phi_{2}=-0.29, \phi_{3}=0.15, \theta_{1}=0.42, \theta_{2}=-0.29, \theta_{3}=0.15$
(a) Indicate whether or not each $\operatorname{ARMA}(p, q)$ model is causal, and indicate whether or not each ARMA $(p, q)$ model is invertible.
i. This model is causal and invertible, it's AR part is the same as a causal AR model in 1. and its MA part is the same as an invertible model in 2 .
ii. This model is causal and invertible, it's AR part is the same as a causal AR model in 1. and its MA part is the same as an invertible model in 2 .
iii. This model is causal and invertible, it's AR part is the same as a causal AR model in 1. and its MA part is the same as an invertible model in 2 .
iv. This model is causal and invertible, it's AR part is the same as a causal AR model in 1. and its MA part is the same as an invertible model in 2 .
v . This model is causal and invertible, it's AR part is the same as a causal AR model in 1. and its MA part is the same as an invertible model in 2.

A note - you might have noticed that at least one of these models has a common root. This is not proper ARMA model form, but it doesn't affect whether or not the model is causal or invertible, nor does it affect the autocorrelation function - it just means that there are multiple ways we could choose the ARMA $(p, q)$ parameters to get the same autocovariance function. That said, no one was penalized for noting the presence of a common root.
(b) Using ARMAacf to compute $\rho_{x}(h)$, plot the autocorrelation function $\rho_{x}(h)$ for $h=0, \ldots, 10$ for the causal and invertible ARMA $(p, q)$ models on a single plot. Include a dotted horizontal line at 0 .

```
pars <- list(c(0.99), c(0.04, 0.92), c(0.04, -0.92), c(0.42, -0.29, 0.15))
lag.max <- 10
acfs <- matrix(nrow = length(pars)^2, ncol = lag.max + 1)
for (i in 1:length(pars)) {
    for (j in 1:length(pars)) {
        acfs[length(pars)*(i - 1) + j, ] <- ARMAacf(ar = pars[[i]], ma = pars[[j]], lag.max = lag.max)
    }
}
acfs <- acfs[c(1, 3, 6, 10, 16), ]
plot(0:lag.max, acfs[1, ], type = "n", ylim = range(acfs, na.rm = TRUE))
for (i in 1:nrow(acfs)) {
    points(0:lag.max, acfs[i, ], col = i, pch = 16)
    lines(0:lag.max, acfs[i, ], col = i)
}
abline(h = 0, lty = 3)
legend("bottomleft", col = 1:5, pch = rep(16, 5), legend = c("i", "ii", "iii", "iv", "v"), bty = "n")
```


(c) Based on the plot you made in (b), describe what kinds of values of the autoregressive and moving average coefficients $\phi_{1}, \ldots, \phi_{p}$ and $\theta_{1}, \ldots, \theta_{q}$ produce the following behaviors, using at most one sentence for each:

- Quickly decaying correlations for small $h$ and slowly decaying correlations for large $h$;
- Periodic/seasonal/cyclical behavior.
- We observe quickly decaying correlations for small $h$ and slowly decaying correlations for large $h$ in models ii., iv., and v. These are all models with a large negative MA component, and at least one moderately large AR component. Based on what we saw in 2. and what we know about MA models, it looks like the MA component only contributes to the first $q$ autocorrelations, and the AR component dominates the autocorrelations when $q>2$.
- We observe periodic/seasonal/cyclical behavior in models iii., iv. and v. In all of these cases, either the AR polynomial has at least one complex root or the AR parameters have switching signs.

