

Introduction to AR, MA, and ARMA Models

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The material in this set of notes is based on S&S Chapter 3, specifically 3.1-3.2. We're finally going to define our first time series model! ☺ The first time series model we will define is the **autoregressive (AR)** model. We will then consider a different simple time series model, the **moving average (MA)** model. Putting both models together to create one more general model will give us the **autoregressive moving average (ARMA)** model.

The AR Model

The first kind of time series model we'll consider is an **autoregressive (AR)** model. This is one of the most intuitive models we'll consider. The basic idea is that we will model the response at time t , y_t , as a linear function of its p previous values and some independent random noise, e.g.

$$y_t = 0.5y_{t-1} + w_t, \tag{1}$$

where y_t is stationary and $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$. This kind of model is especially well suited to forecasting, as

$$\mathbb{E}[y_{t+1}|y_t] = 0.5y_{t-1}. \tag{2}$$

We explicitly define an **autoregressive model of order p** , abbreviated as **AR** (p) as:

$$(y_t - \mu) = \phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + \cdots + \phi_p (y_{t-p} - \mu) + w_t, \quad (3)$$

where $\phi_p \neq 0$, y_t is stationary with mean $\mathbb{E}[y_t] = \mu$, and $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$. For convenience:

- We'll often assume $\mu = 0$, so

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t. \quad (4)$$

- We'll introduce the **autoregressive operator** notation:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p, \quad (5)$$

where $B^p y_t = y_{t-p}$ is the **backshift operator**. This allows us to rewrite (3) and (4) more concisely as $\phi(B)(y_t - \mu) = w_t$ and

$$\phi(B)(y_t) = w_t, \quad (6)$$

respectively.

An **AR** (p) model looks like a linear regression model, but the covariates are also random variables. We'll start building an understanding of the **AR** (p) model by starting with the simpler special case where $p = 1$.

The **AR** (1) model with $\mu = 0$ is a special case of (3)

$$y_t = \phi_1 y_{t-1} + w_t. \quad (7)$$

A natural thing to do is to try to rewrite y_t as a function of ϕ_1 and the previous values of the random errors. Then (7) will look more like a classical regression problem, as it will no longer have random variables as covariates. Furthermore, if we could rewrite y_t as a function of ϕ_1 and the random errors \mathbf{w} , then y_t would be a **causal linear process**.

A **causal linear process** y_t is defined to be a linear combination of white noise w_t and

is given by

$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j w_{t-j},$$

where the coefficients satisfy $\sum_{j=0}^{\infty} |\psi_j| < \infty$, w_t are independent and identically distributed with mean 0 and variance σ_w^2 , and $\mu = \mathbb{E}[y_t] < \infty$. This is called a causal linear process because y_t only depends on past and present values of white noise, i.e. we can think of the present and past values of the white noise $w_t, w_{t-1}, w_{t-2}, \dots$ as causing the variability we observe in y_t . The condition $\sum_{j=0}^{\infty} |\psi_j| < \infty$ ensures that each y_t has finite variance $\mathbb{V}[y_t] = \sigma_w^2 \sum_{j=0}^{\infty} \psi_j^2 < \infty$, the same probability distribution, and decaying autocorrelations $\rho_y(h) \rightarrow 0$ as $h \rightarrow \infty$. Importantly, it can be shown that the autocovariance function of a causal linear process is

$$\gamma_y(h) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_{j+h} \psi_j, \quad (8)$$

for $h \geq 0$, recalling that $\gamma_y(h) = \gamma_y(-h)$. This also means that once we know the linear process representation of any time series process, we can easily compute its autocovariance (and autocorrelation) functions. We will often use the **infinite moving average operator** shorthand $1 + \psi_1 B + \psi_2 B^2 + \dots \psi_j B^j + \dots = \psi(B)$.

We can start rewriting y_t as follows:

$$\begin{aligned} y_t &= \phi_1^2 y_{t-1} + \phi_1 w_{t-1} + w_t \\ &= \phi_1^3 y_{t-2} + \phi_1^2 w_{t-2} + \phi_1 w_{t-1} + w_t \\ &= \underbrace{\phi_1^k y_{t-k}}_{(*)} + \sum_{j=0}^{k-1} \phi_1^j w_{t-j}. \end{aligned}$$

We can see that we can almost take the lagged values of \mathbf{y} out of the right hand side. Fortunately, when $|\phi_1| < 1$, then

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[\left(y_t - \sum_{j=0}^{k-1} \phi_1^j w_{t-j} \right)^2 \right] = \lim_{k \rightarrow \infty} \phi^{2k} \mathbb{E} [y_{t-k}^2] = 0,$$

because $\mathbb{E}[y_{t-k}^2]$ is constant as long as y_t is stationary is assumed. This means that when $|\phi_1| < 1$, then we can write elements of the response y_t as a linear function the previous values of the random errors:

$$y_t = \sum_{j=0}^{\infty} \phi_1^j w_{t-j}. \quad (9)$$

(9) is the **causal linear process** representation of an **AR(1)** model. It follows that the autocovariance function

$$\begin{aligned} \gamma_y(h) &= \sigma_w^2 \sum_{j=0}^{\infty} \phi_1^{j+h} \phi_1^j \\ &= \sigma_w^2 \phi_1^h \sum_{j=0}^{\infty} \phi_1^{2j} \\ &= \sigma_w^2 \phi_1^h \left(\frac{1}{1 - \phi_1^2} \right). \end{aligned} \quad (10)$$

and the autocorrelation function is

$$\rho_y(h) = \phi_1^h. \quad (11)$$

Note that this is not the only way to compute the values of autocovariance function. We could compute them directly from (4),

$$\begin{aligned} \gamma_y(h) &= \mathbb{E}[y_{t-h}y_t] \\ &= \mathbb{E}[y_{t-h}(\phi_1 y_{t-1} + w_t)] \\ &= \phi_1 \mathbb{E}[y_{t-1-(h-1)}y_{t-1}] + \mathbb{E}[y_{t-h}w_t] \\ &= \phi_1 \gamma_y(h-1). \end{aligned} \quad (12)$$

This gives us a recursive relation that we can use to compute the autocovariance function

$\gamma_y(h)$, starting from $\gamma_y(0)$. We can compute $\gamma_y(0)$ using substitution:

$$\begin{aligned}
\gamma_y(0) &= \mathbb{E} [y_t^2] && (13) \\
&= \mathbb{E} [y_t^2] \\
&= \mathbb{E} [(\phi_1 y_{t-1} + w_t)^2] \\
&= \mathbb{E} [\phi_1^2 y_{t-1}^2 + 2\phi_1 w_t y_{t-1} + w_t^2] \\
&= \phi_1^2 \mathbb{E} [y_{t-1}^2] + \sigma_w^2 \\
&= \sigma_w^2 \sum_{j=0}^{\infty} \phi_1^{2j} && \text{(follows from continued substitution)} \\
&= \frac{\sigma_w^2}{1 - \phi_1^2}, && \text{if } |\phi_1| < 1, \gamma_y(0) = \infty \text{ otherwise!}
\end{aligned}$$

If $|\phi_1| < 1$, then it is easy to see that the **AR**(1) model y_t is stationary because the mean of each y_t is zero and the autocovariance function $\gamma_y(h) = \sigma_w^2 \phi_1^h \left(\frac{1}{1 - \phi_1^2} \right)$ depends only on the lag, h , and is finite when $h = 0$, $\gamma_y(0) < \infty$. What happens when $|\phi_1| > 1$? (9) does **not** have a causal linear process representation if $|\phi_1| > 1$, and because $\gamma_y(0) = \sum_{j=0}^{\infty} |\phi_1^j| = +\infty$, so y_t will *not* be a stationary process because it will not have finite variance.

Understanding when a **AR**(p) model is stationary is more difficult than understanding when an **AR**(1) model is stationary. We figured out when an **AR**(1) model is stationary by finding the coefficients ψ_1, \dots, ψ_j of its causal linear process representation as a function of the AR coefficient ϕ_1 , and showing that the squared sum of the coefficients $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

The linear process representation is especially useful for an **AR**(p) model when $p > 1$, because computing the autocovariance function $\gamma_y(h)$ directly as we did in (12) and (13) gets much more cumbersome when $p > 1$. We can see this in the **AR**(2) case, where we have

$$y_t = \phi_2 y_{t-2} + \phi_1 y_{t-1} + w_t. \quad (14)$$

We can get a recursive relation for the autocovariance function $\gamma_y(h)$ starting from $\gamma_y(0)$

and $\gamma_y(1)$ as follows:

$$\begin{aligned}
\gamma_y(h) &= \mathbb{E}[y_{t-h}y_t] \\
&= \mathbb{E}[y_{t-h}(\phi_1y_{t-1} + \phi_2y_{t-2} + w_t)] \\
&= \phi_1\mathbb{E}[y_{t-1-(h-1)}y_{t-1}] + \phi_2\mathbb{E}[y_{t-2-(h-2)}y_{t-2}] + \mathbb{E}[y_{t-h}w_t] \\
&= \phi_1\gamma_y(h-1) + \phi_2\gamma_y(h-2).
\end{aligned}$$

We can try to compute $\gamma_y(0)$ and $\gamma_y(1)$ using substitution:

$$\begin{aligned}
\gamma_y(0) &= \mathbb{E}[y_t^2] \\
&= \mathbb{E}[(\phi_1y_{t-1} + \phi_2y_{t-2} + w_t)^2] \\
&= \mathbb{E}[\phi_1^2y_{t-1}^2 + \phi_2^2y_{t-2}^2 + 2\phi_1\phi_2y_{t-1}y_{t-2} + 2\phi_1y_{t-1}w_t + 2\phi_2y_{t-2}w_t + w_t^2] \\
&= \mathbb{E}[\phi_1^2y_{t-1}^2 + \phi_2^2y_{t-2}^2 + 2\phi_1\phi_2y_{t-1}y_{t-2}] + \sigma_w^2 \\
&= (\phi_1^2 + \phi_2^2)\gamma_y(0) + 2\phi_1\phi_2\gamma_y(1) + \sigma_w^2.
\end{aligned}$$

$$\begin{aligned}
\gamma_y(1) &= \mathbb{E}[y_{t-1}y_t] \\
&= \mathbb{E}[y_{t-1}(\phi_1y_{t-1} + \phi_2y_{t-2} + w_t)] \\
&= \mathbb{E}[\phi_1y_{t-1}^2 + \phi_2y_{t-1}y_{t-2} + y_{t-1}w_t] \\
&= \phi_1\gamma_y(0) + \phi_2\gamma_y(1) \\
\gamma_y(1) &= \left(\frac{\phi_1}{1-\phi_2}\right)\gamma_y(0).
\end{aligned}$$

$$\begin{aligned}
\gamma_y(0) &= (\phi_1^2 + \phi_2^2) \gamma_y(0) + 2(\phi_1\phi_2) \left(\frac{\phi_1}{1-\phi_2} \right) \gamma_y(0) + \sigma_w^2 \\
\left(\frac{1 - \phi_2 - \phi_1^2 - \phi_2^2 + \phi_2^3 - \phi_1^2\phi_2}{1 - \phi_2} \right) \gamma_y(0) &= \sigma_w^2 \\
\gamma_y(0) &= \sigma_w^2 \left(\frac{1 - \phi_2}{1 - \phi_2 - \phi_1^2 - \phi_2^2 + \phi_2^3 - \phi_1^2\phi_2} \right) \\
\gamma_y(1) &= \sigma_w^2 \left(\frac{\phi_1}{1 - \phi_2 - \phi_1^2 - \phi_2^2 + \phi_2^3 - \phi_1^2\phi_2} \right)
\end{aligned}$$

Clearly, this gets *very* complicated, even though we only have two lags!

Unfortunately, it's much harder to find the linear process representation of an **AR**(p) model by simple substitution as we did with an **AR**(1) model. Substituting according to (14)

$$\begin{aligned}
y_t &= \phi_1\phi_2 y_{t-3} + (\phi_2 + \phi_1^2) y_{t-2} + \phi_1 w_{t-1} + w_t \\
&= (\phi_2 + \phi_1^2) \phi_2 y_{t-4} + \phi_1 (2\phi_2 + \phi_1^2) y_{t-3} + (\phi_2 + \phi_1^2) w_{t-2} + \phi_1 w_{t-1} + w_t \\
&= (\phi_2 + \phi_1^2) \phi_2 y_{t-4} + \phi_1 (2\phi_2 + \phi_1^2) (\phi_2 y_{t-5} + \phi_1 y_{t-4} + w_{t-3}) + (\phi_2 + \phi_1^2) w_{t-2} + \phi_1 w_{t-1} + w_t \\
&= \phi_1\phi_2 (2\phi_2 + \phi_1^2) y_{t-5} + (\phi_2^2 + \phi_1^2\phi_2 + 2\phi_1\phi_2^2 + \phi_1^3\phi_2) y_{t-4} + \\
&\quad \phi_1 (2\phi_2 + \phi_1^2) w_{t-3} + (\phi_2 + \phi_1^2) w_{t-2} + \phi_1 w_{t-1} + w_t \dots
\end{aligned}$$

Again, this is *not* working out nicely!

Instead, we can find the values of $\psi_1, \dots, \psi_j, \dots$ that satisfy $\phi(B)\psi(B)w_t = w_t$, which follows from substituting $y_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$ into (20). This is equivalent to finding the inverse function $\phi^{-1}(B)$ that satisfies $\phi(B)\phi^{-1}(B)w_t = w_t$.

We can see how this method for finding the values of $\psi_1, \dots, \psi_j, \dots$ works by returning to the **AR**(1) case. The values $\psi_1, \dots, \psi_j, \dots$ that satisfy $\phi(B)\psi(B)w_t = w_t$ solve:

$$1 + (\psi_1 - \phi_1)B + (\psi_2 - \psi_1\phi_1)B^2 + \dots + \psi_j B^j + \dots = 1, \quad (15)$$

where (15) follows from expanding $\phi(B)$ and $\psi(B)$. This allows us to recover the linear

process representation of the **AR**(1) process in a different way, as (15) holds if all of the coefficients for B^j with $j > 0$ are equal to zero, i.e. $\psi_k - \psi_{k-1}\phi_1 = 0$ for $k > 1$.

Now let's try this approach for the **AR**(2) case. We have

$$\begin{aligned} 1 &= (1 - \phi_1 B - \phi_2 B^2) (1 + \psi_1 B + \psi_2 B^2 + \dots + \psi_j B^j + \dots) \\ &= 1 + (\psi_1 - \phi_1) B + (\psi_2 - \phi_2 - \phi_1 \psi_1) B^2 + (\psi_3 - \phi_1 \psi_2 - \phi_2 \psi_1) B^3 + \dots + \\ &\quad (\psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2}) B^j + \dots \end{aligned}$$

We see that we can compute the values of $\psi_1, \dots, \psi_j, \dots$ recursively,

$$\begin{aligned} \psi_1 &= \phi_1 \\ \psi_2 &= \phi_2 + \phi_1^2 \\ \psi_3 &= \phi_1 (\phi_2 + \phi_1^2) + \phi_2 \phi_1 \\ &= \phi_1 (2\phi_2 + \phi_1^2) \\ &= 2\phi_1 \phi_2 + \phi_1^3 \\ \psi_4 &= \phi_1^2 (2\phi_2 + \phi_1^2) + \phi_2 (\phi_2 + \phi_1^2) \\ &= 3\phi_1^2 \phi_2 + \phi_2^2 + \phi_1^4 \\ \psi_5 &= \phi_1^3 (2\phi_2 + \phi_1^2) + \phi_1 \phi_2 (\phi_2 + \phi_1^2) + \phi_1 \phi_2 (2\phi_2 + \phi_1^2) \\ &= 3\phi_1^3 \phi_2 + \phi_1 \phi_2^2 + \phi_1^5 + 2\phi_1 \phi_2^2 + \phi_1^3 \phi_2 \\ &= 4\phi_1^3 \phi_2 + 3\phi_1 \phi_2^2 + \phi_1^5 \\ \psi_6 &= 4\phi_1^4 \phi_2 + 3\phi_1^2 \phi_2^2 + \phi_1^6 + 3\phi_1^2 \phi_2^2 + \phi_2^3 + \phi_1^4 \phi_2 \\ &= 5\phi_1^4 \phi_2 + 6\phi_1^2 \phi_2^2 + \phi_1^6 + \phi_2^3 \end{aligned}$$

and so on. Again, this is not working out nicely. In general, we won't work with closed form expressions for the autocovariance function of an **AR**(p) model with $p > 1$.

It's also very tricky to figure out when **AR**(p) model is **stationary** for $p > 1$. An **AR**(p)

model is **stationary** for $p > 1$ when all of the roots of the **AR polynomial**

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p,$$

lie outside the unit circle, i.e. $\phi(z) \neq 0$ for $|z| \leq 1$. This condition ensures that the $\sum_{j=1}^{\infty} |\psi_j| < \infty$. This is not very intuitive. If we want to try to get a handle on why the roots of the AR polynomial need to lie outside the unit circle for a **AR(p)** model to be **stationary**, we need to take a look at the proof. You won't be tested on your understanding of this - we'll just go through it here in case you are curious following along the proof of Theorem 3.2 in Chan (2010).

Let's suppose that $\phi(z)$ has roots r_1, \dots, r_p that satisfy $1 < |r_1| \leq \dots \leq |r_p|$, i.e. $\phi(r_j) = 0$ for $j = 1, \dots, p$. Then this ensures that we can invert $\phi(z)$ when $z \leq |r_1|$. Recalling that $\psi(B)$ can be thought of as the inverse of $\phi(B)$, this means that

$$\frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j < \infty \text{ if } |z| \leq |r_1|,$$

where $\psi_0 = 1$. Then we can invert $\phi(z)$ at any value of $z < |r_1|$, e.g. at $z = 1 + \delta < |r_1|$, where $\delta > 0$. Writing this out, we have

$$\frac{1}{\phi(1 + \delta)} = \sum_{j=0}^{\infty} \psi_j (1 + \delta)^j < \infty. \tag{16}$$

If (16), then there must be some constant $M > 0$ that gives an upper bound for all $|\psi_j (1 + \delta)^j|$, i.e. $|\psi_j (1 + \delta)^j| \leq M$ for all $j = 0, 1, 2, \dots$. Shifting things around, this is

equivalent to $|\psi_j| \leq M(1 + \delta)^{-j}$. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} |\psi_j| &\leq M \sum_{j=1}^{\infty} \left(\frac{1}{1 + \delta}\right)^j \\
&= M \left(\sum_{j=0}^{\infty} \left(\frac{1}{1 + \delta}\right)^j - 1 \right) \\
&= M \left(\frac{1}{1 - \frac{1}{1 + \delta}} - 1 \right) && \text{(follows from } \frac{1}{1 + \delta} < 1 \text{ if } \delta > 0) \\
&= M \left(\frac{1 + \delta}{1 + \delta - 1} - 1 \right) = M \left(\frac{1}{\delta} \right) < \infty.
\end{aligned}$$

The MA Model

Instead of assuming that elements of a time series y_t are linear function of previous elements of the time series y_1, \dots, y_{t-1} and independent, identically distributed noise w_t , we might assume that elements of a time series y_t are a linear function of all of the current and previous noise variates, w_1, \dots, w_{t-1} . The latter gives us the **moving average model of order q** , abbreviated as **MA (q)**. The **MA (q)** model is explicitly defined as

$$y_t - \mu = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots + \theta_q w_{t-q}, \quad (17)$$

where $\theta_q \neq 0$, $\mathbb{E}[y_t] = \mu$, and $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$. For convenience:

- We'll often assume $\mu = 0$, so

$$y_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots + \theta_q w_{t-q}. \quad (18)$$

- We'll introduce the **moving average operator** notation:

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_p B^p, \quad (19)$$

which allows us to rewrite (17) and (18) more concisely as $y_t - \mu = \theta(B) w_t$ and

$$y_t = \theta(B) w_t, \quad (20)$$

respectively.

Again, the $\mathbf{MA}(q)$ model looks like a linear regression model. Importantly, the $\mathbf{MA}(q)$ model is stationary for any values of the parameters $\theta_1, \dots, \theta_q$.

Like we did with the $\mathbf{AR}(p)$ model, we'll start building an understanding of the $\mathbf{MA}(q)$ by starting with the simpler special case where $q = 1$,

$$y_t = \theta_1 w_{t-1} + w_t. \quad (21)$$

It is easy to see that this $\mathbf{MA}(q)$ model is mean zero. We can compute the autocovariance function as follows:

$$\begin{aligned} \gamma_y(h) &= \mathbb{E}[y_t y_{t-h}] \\ &= \mathbb{E}[(\theta_1 w_{t-1} + w_t)(\theta_1 w_{t-h-1} + w_{t-h})] \\ &= \mathbb{E}[\theta_1^2 w_{t-1} w_{t-h-1} + \theta_1 w_t w_{t-h-1} + \theta_1 w_{t-1} w_{t-h} + w_t w_{t-h}] \\ &= \mathbb{E}[\theta_1^2 w_{t-1} w_{t-h-1} + \theta_1 w_{t-1} w_{t-h} + w_t w_{t-h}] \\ &= \begin{cases} \sigma_w^2 (\theta_1^2 + 1) & h = 0 \\ \theta_1 & h = 1 \\ 0 & h > 1 \end{cases}. \end{aligned} \quad (22)$$

The corresponding autocorrelation function is

$$\rho_y(h) = \begin{cases} \frac{\theta_1}{\theta_1^2 + 1} & h = 1 \\ 0 & h > 1 \end{cases}. \quad (23)$$

The autocovariance and autocorrelation functions of the $\mathbf{MA}(q)$ model are noteworthy in two ways:

- (•) The autocorrelation function $\rho_y(h)$ is bounded, $\rho_y(h) \leq 1/2$ for $h = 1$.
- (*) The parameters of the $\mathbf{MA}(q)$ model do not uniquely determine the autocovariance and autocorrelation function values. θ_1 and σ_w^2 do not uniquely determine the value

of the autocovariance function $\gamma_y(h)$, and θ_1 does not determine the value of the autocorrelation function.

It is easiest to understand (*) via some examples. First, we compute $\gamma_y(h)$ and $\rho_y(h)$ for a **MA**(1) process with $\theta_1 = 5$ and $\sigma_w^2 = 1$,

$$\gamma_y(h) = \begin{cases} 5^2 + 1 = 26 & h = 0 \\ 5 & h = 1 \\ 0 & h > 1 \end{cases} \quad \text{and} \quad \rho_y(h) = \begin{cases} \frac{5}{5^2+1} = \frac{5}{26} & h = 1 \\ 0 & h > 1 \end{cases} .$$

Compare this to $\gamma_y(h)$ and $\rho_y(h)$ for a **MA**(1) process with $\theta_1 = 1/5$ and $\sigma_w^2 = 25$,

$$\gamma_y(h) = \begin{cases} 25 \left(\frac{1}{5^2} + 1 \right) = 25 \left(\frac{1+25}{25} \right) = 26 & h = 0 \\ 25 \left(\frac{1}{5} \right) = 5 & h = 1 \\ 0 & h > 1 \end{cases} \quad \text{and} \quad \rho_y(h) = \begin{cases} \frac{\frac{1}{5}}{\frac{1}{5^2}+1} = \frac{5}{26} & h = 1 \\ 0 & h > 1 \end{cases} .$$

Both sets of **MA**(1) parameters give the values of the autocovariance and autocorrelation functions! This is undesirable - it means that even if we know that our time series is mean zero with a specific autocovariance function $\gamma_y(h)$ autocorrelation function $\rho_y(h)$, we can't find a **unique** pair of corresponding **MA**(1) parameter values (θ_1, σ_w^2) . ☹

We solve this problem by requiring that our **MA**(1) model be **invertible**, which means that it has an infinite autoregressive representation $(1 + \pi_1 B + \pi_2 B^2 + \dots + \pi_j B^j + \dots) y_t = w_t$ with $\sum_{j=1}^{\infty} |\pi_j| < \infty$. We can find a **unique** pair of corresponding **MA**(1) parameter values (θ_1, σ_w^2) if we restrict our attention to the parameter values that give an **invertible** **MA**(1) model. What we mean by this is that we can rearrange (21) to resemble a **AR**(1)

model for w_t ,

$$\begin{aligned}
w_t &= -\theta_1 w_{t-1} + y_t \\
&= \theta_1^2 w_{t-2} - \theta_1 y_{t-1} + y_t \\
&= -\theta_1^3 w_{t-3} + \theta_1^2 y_{t-2} - \theta_1 y_{t-1} + y_t \\
&= (-\theta_1)^k w_{t-k} + \sum_{j=0}^k (-\theta_1)^j y_{t-j},
\end{aligned}$$

where $\lim_{k \rightarrow \infty} (-\theta_1)^k w_{t-k} + \sum_{j=0}^k (-\theta_1)^j y_{t-j} = \sum_{j=0}^{\infty} (-\theta_1)^j y_{t-j}$. Recalling the **AR**(1) model, this will be the case when $|\theta_1| < 1$. Going back to our example where we considered the **MA**(1) parameters $(\theta_1, \sigma_w^2) = (5, 1)$ and $(\theta_1, \sigma_w^2) = (\frac{1}{5}, 25)$, this means that only the latter pair $(\theta_1, \sigma_w^2) = (\frac{1}{5}, 25)$ satisfy our definition of a **MA**(1) model.

More generally, requiring that an **MA**(q) model be **invertible** ensures that we can find a **unique** set of corresponding **MA**(q) parameter values $(\theta_1, \dots, \theta_q, \sigma_w^2)$ if we know that our time series is **MA**(q) with mean zero, a specific autocovariance function $\gamma_y(h)$, and autocorrelation function $\rho_y(h)$. We introduce some additional notation for this; an **MA**(q) model is **invertible** if we can write $w_t = \pi(B) y_t$, where $\pi(B) = 1 + \pi_1 B + \dots + \pi_j B^j + \dots$ and $\sum_{j=0}^{\infty} |\pi_j| < \infty$. This looks a lot like the problem of ensuring that a **AR**(p) model is **stationary**, and it turns out that an **MA**(q) model is **invertible** if when all of the roots of the **MA polynomial**

$$\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q,$$

lie outside the unit circle, i.e. $\theta(z) \neq 0$ for $|z| \leq 1$.

The ARMA Model

The **autoregressive moving average (ARMA)** model combines the **AR** and **MA** models.

We define an **ARMA**(p, q) model as:

$$(y_t - \mu) = \phi_1 (y_{t-1} - \mu) + \cdots + \phi_p (y_{t-p} - \mu) + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q} + w_t, \quad (24)$$

where $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$, y_t is stationary, $\phi_p \neq 0$, $\theta_q \neq 0$, $\sigma_w^2 > 0$, and the MA and AR polynomials $\theta(B)$ and $\phi(B)$ have no common roots. We refer to p as the **autoregressive order** and q as the **moving average order**. Again, for convenience we will usually assume $\mu = 0$, so

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}. \quad (25)$$

Using operator notation becomes especially beneficial for **ARMA**(p, q) models; we can just write $\phi(B) y_t = \theta(B) w_t$ instead of (25). Note that:

- Setting $p = 0$ gives a **MA**(q) model;
- Setting $q = 0$ gives an **AR**(p).

As with **AR**(p) and **MA**(q) models, we will need to figure out when an **ARMA**(p, q) is **stationary** and **invertible**. Fortunately, this is simple given the work we've already done for **MA**(q) and **AR**(p) models. An **ARMA**(p, q) is:

- **Stationary**, i.e. we can find $\psi_1, \dots, \psi_j, \dots$ such that $\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}$ that satisfy $\sum_{j=0}^{\infty} |\psi_j| < \infty$ for $|z| < 1$, if $\phi(z) \neq 0$ for $|z| \leq 1$;
- **Invertible**, i.e. we can find $\pi_1, \dots, \pi_j, \dots$ such that $\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)}$ that satisfy $\sum_{j=0}^{\infty} |\pi_j| < \infty$ for $|z| < 1$, if $\theta(z) \neq 0$ for $|z| \leq 1$.

Returning to the definition of an **ARMA**(p, q) model, it is not immediately obvious why we require that the moving average and autoregressive polynomials $\theta(B)$ and $\phi(B)$ have no

common roots. Consider the following model, which resembles an **ARMA** (p, q) model:

$$y_t = 0.5y_{t-1} - 0.5w_{t-1} + w_t, \quad (26)$$

where y_t is stationary and $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$. It's easy to see that the mean function $\mu = 0$. The autocovariance function $\gamma_y(h)$ satisfies:

$$\begin{aligned} \gamma_y(h) &= \mathbb{E}[y_t y_{t-h}] \\ &= \mathbb{E}[(0.5y_{t-1} - 0.5w_{t-1} + w_t) y_{t-h}] \\ &= 0.5\mathbb{E}[y_{t-1} y_{t-h}] - 0.5\mathbb{E}[w_{t-1} y_{t-h}] + \mathbb{E}[w_t y_{t-h}] \\ &= \begin{cases} 0.5\gamma_y(0) - 0.5\sigma_w^2 & h = 1 \\ 0.5\gamma_y(h-1) & h > 1 \end{cases} \end{aligned} \quad (27)$$

We just need to combine this with a starting value, $\gamma_y(0)$:

$$\begin{aligned} \gamma_y(0) &= \mathbb{E}[y_t^2] \\ &= \mathbb{E}[0.5^2 y_{t-1}^2 + 0.5^2 w_{t-1}^2 + w_t^2 - (2)(0.5)^2 w_{t-1}^2] \\ &= 0.5^2 \gamma_y(0) + (1 - 0.5^2) \sigma_w^2 \implies \gamma_y(0) = \sigma_w^2 \end{aligned}$$

Plugging this in to (27), for $h > 0$ we get

$$\gamma_y(h) = 0!$$

This means that (26) is equivalent to the white noise model, $y_t = w_t$!

If we examine the corresponding AR and MA polynomials, we see that they share the common factor $1 - 0.5B$, $\theta(B) = 1 - 0.5B$ and $\phi(B) = 1 - 0.5B$. Dividing each by the common factor yields $\theta(B) = 1$ and $\phi(B) = 1$, which gives us the familiar definition of the white noise model, $y_t = w_t$. This is why we require that the the moving average and autoregressive polynomials $\theta(B)$ and $\phi(B)$ have no common roots, otherwise we could mistake a white noise process for an **ARMA** (p, q) process with $p, q > 0$.

As with the **AR** (p) model, the linear process representation of an **ARMA** (p, q) model

is especially useful for computing the autocovariance function of an **ARMA** (p, q) model. Using the same approach we used for the **AR** (p) model, the values of $\psi_1, \dots, \psi_j, \dots$ that satisfy $y_t = \psi(B) w_t$ with $\sum_{j=0}^{\infty} |\psi_j| < \infty$ can be computed by substituting $\psi(B) w_t$ into the equation that defines the **ARMA** (p, q) model, $\phi(B) y_t$, and matching the coefficients for each power of B on each side, i.e.

$$\begin{aligned} \phi(B) \psi(B) w_t &= \theta_z w_t \\ \implies (1 - \phi_1 B - \dots - \phi_p B^p) (1 + \psi_1 B + \dots + \psi_j B^j) w_t &= (1 + \theta_1 B + \dots + \theta_q B^q) w_t. \end{aligned}$$

This yields a sequence of equations that would start with

$$\begin{aligned} \psi_1 - \phi_1 &= \theta_1 \\ \psi_2 - \phi_2 - \phi_1 \psi_1 &= \theta_2, \end{aligned}$$

and continue on for $\psi_3, \dots, \psi_j, \dots$. We will *not* be computing $\psi_1, \dots, \psi_j, \dots$ by hand in class - this requires a knowledge of differential equations that goes above and beyond the prerequisites for this course. However, statistical software like **R** will often include functions that can be used to compute the ψ_1, \dots, ψ_K for some user specified value $K > 1$ given values for ϕ_1, \dots, ϕ_p and $\theta_1, \dots, \theta_p$.